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Error Bounds for Solving Pseudodifferential Equations on Spheres by Collocation with Zonal Kernels

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The problem of solving pseudodifferential equations on spheres by collocation with zonal kernels is considered and bounds for the approximation error are established. The bounds are given in terms of the maximum separation distance of the collocation points, the order of the pseudodifferential operator, and the smoothness of the employed zonal kernel. A by-product of the results is an improvement on the previously known convergence order estimates for Lagrange interpolation.

Key Words: pseudodifferential equation; collocation; zonal kernel; interpolation; approximation order; sphere; positive definite function; radial basis function.

1. INTRODUCTION

Data fitting and solving differential and integral equations on the sphere are areas of growing interest with applications to physical geodesy, potential theory, oceanography, and meteorology [6, 10]. As more and more satellites are being launched into space, the acquisition of global data is becoming more important and more widespread, and the demand for spherical data processing and solving problems of a global nature is increasing.

In this paper we investigate the solution of pseudodifferential equations on spheres by collocation at scattered data points with zonal kernels. Denoting by S^{m-1} the unit sphere in $\mathbb{R}^m (m \geqslant 2)$, a function $\kappa: S^{m-1} \times S^{m-1} \to \mathbb{R}$ is called a *zonal kernel* if it has the form

$$\kappa(p, q) = \psi_\kappa(p \cdot q),$$

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where ψ_κ is a univariate function defined on $[-1, 1]$, called the *shape function*, and $p \cdot q$ is the Euclidean dot product of points $p, q \in S^{m-1}$. Since for a fixed p the value of $\kappa(p, q)$ depends only on the geodesic distance of p from q, the function $\kappa(p, \cdot)$ is radially symmetric with respect to the point p. For this reason $\kappa(p, \cdot)$ is often called a *spherical radial basis function*.

Differential or, more generally, pseudodifferential equations arise in many areas of earth sciences (see, e.g., [10, 25] for many important examples). Given a pseudodifferential operator L and a spherical function f, our objective in this paper is to discuss approximate solutions of the equation

$$Lu = f.$$

In order that this equation may be uniquely solvable, side conditions will be imposed of the form

$$\gamma u = d_\gamma, \qquad \gamma \in \Gamma,$$

where Γ is a collection of linear functionals and $d_\gamma \in \mathbb{R}$ are given "data values." The particular method we study here seeks an approximation of the solution u in the form

$$s_u = s_0 + \sum_{q \in Q} c_q L^q \kappa(\cdot, q), \qquad c_q \in \mathbb{R}, \tag{1}$$

where Q is a finite set of points in S^{m-1}, called the *collocation points*, and s_0 is a linear combination of spherical harmonics that are annihilated by L. Throughout, we follow the convention that $L^q \kappa(\cdot, q)$ means that L is applied to $\kappa(\cdot, q)$ as a function of q. Note that in (1) we abused the notation slightly in that, strictly speaking, $L^q \kappa(\cdot, q) = L^p \kappa(\cdot, p)|_{p=q}$. To determine s_u, that is, to find s_0 and the coefficients c_q, $q \in Q$, we can solve the linear system of equations

$$\begin{aligned} Ls_u(p) &= f(p), \qquad && p \in Q, \\ \gamma s_u &= d_\gamma, && \gamma \in \Gamma. \end{aligned} \tag{2}$$

We consider s_u of the form (1) to ensure that the linear system (2) will be positive semi-definite (see also Remark 17).

The above collocation method is an example of the emerging meshless techniques for solving differential equations, since it only requires the information about the location of the points Q, which need not be interconnected. Meshless methods are a challenging topic with the potential to become a feasible alternative to finite element methods [27].

In this paper we will furnish convergence order estimates for the collocation problem explicitly in terms of a mesh norm h that measures the density of the points Q. The crux of our approach is to transform the collocation problem to a Lagrange interpolation problem. In particular, defining a new kernel

$$\kappa_L(p, q) := L^p L^q \kappa(p, q),$$

the first set of equations in (2) is equivalent to finding a function of the form

$$s_f := \sum_{q \in Q} c_q \kappa_L(\cdot, q),$$

satisfying the interpolation conditions

$$s_f(p) = f(p), \qquad p \in Q.$$

There are many known error estimates for this type of Lagrange interpolation in the literature, including [9, 12, 14, 26]. For our purposes, the approach taken in [14] will prove most useful. In addition, the work presented here also draws on some ideas from other recent sources on interpolation in $\mathbb{R}^m$ [18, 23], solving differential equations in $\mathbb{R}^m$ using collocation [4, 7, 8], and spherical interpolation and approximation [5, 10, 17]. The system of Eqs. (2) can be interpreted as a generalized Hermite interpolation problem. Generalized Hermite interpolation on spheres and on other differentiable manifolds has been studied in [1, 2, 22].

The analysis of the error $u - s_u$ will be done in Sobolev spaces and in certain Hilbert spaces called native spaces. Our final estimates, formulated in Theorem 16, will be of the form

$$\|u - s_u\| \leqslant C(h) \|f\|,$$

where $C(h)$ is a function converging to zero as $h \to 0$, whose rate of convergence depends on the smoothness of the zonal kernel κ and the order of the pseudodifferential operator L, and where the norms $\|\cdot\|$ are taken in appropriate function spaces.

The layout of the paper is as follows. In Section 2 we introduce zonal kernels and their associated native spaces. Section 3 describes an abstract interpolation problem that will be useful in formulating the collocation problem in Section 4. The analysis of the convergence order of the collocation method will be carried out in Sections 5 and 6.

In the remainder of this section we recall some standard definitions and tools needed for analysis of spherical functions. For a more detailed account, the reader is referred to [21].

Let Δ^* be the *Laplace–Beltrami operator* on S^{m-1}, which can be defined for all sufficiently smooth functions u on S^{m-1} as

$$\Delta^* u(p) := \Delta v(p), \qquad p \in S^{m-1},$$

where Δ is the Laplacian in $\mathbb{R}^m$ and v is the homogeneous extension of degree zero of u to $\mathbb{R}^m$, or $v(p) := u(p/\|p\|)$, $p \in \mathbb{R}^m \setminus \{0\}$. The eigenvalues for the eigenvalue problem

$$\Delta^* u + \lambda u = 0$$

are $\lambda_k = k(k+m-2)$, where k is a nonnegative integer. The space of spherical harmonics of degree k consists of all infinitely differentiable functions that are eigenfunctions of Δ^* corresponding to λ_k. This space has dimension $N(k)$, where

$$N(0) = 1, \qquad \text{and} \qquad N(k) = \frac{2k+m-2}{k} \binom{k+m-3}{k-1}, \qquad k > 0.$$

Note that $N(k) = O(k^{m-2})$. Given an orthonormal basis $\{Y_{kl} : l = 1, ..., N(k)\}$ for the space of spherical harmonics of degree k, the collection

$$\{Y_{kl} : l = 1, ..., N(k), k \geqslant 0\}$$

forms an orthonormal basis for $L_2 := L_2(S^{m-1})$. According to the well-known Addition Theorem,

$$\sum_{l=1}^{N(k)} Y_{kl}(p)\, Y_{kl}(q) = \omega^{-1} N(k)\, P_k(p \cdot q), \qquad p, q \in S^{m-1}, \qquad k \geqslant 0, \tag{3}$$

where P_k is the Legendre polynomial of degree k in m dimensions, normalized such that $P_k(1) = 1$, and ω is the surface area of S^{m-1}. Legendre polynomials satisfy the inequality

$$|P_k(t)| \leqslant 1, \qquad t \in [-1, 1], \qquad k \geqslant 0. \tag{4}$$

Spherical harmonics can be used to give a Fourier analysis on the sphere. In particular, every function $u \in L_2$ has an associated Fourier series

$$u = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} \hat{u}_{kl} Y_{kl},$$

where the equality holds in L_2. The Fourier coefficients $\hat{u}_{kl} \in \mathbb{R}$ are obtained via

$$\hat{u}_{kl} = \int_{S^{m-1}} u(p)\, Y_{kl}(p)\, dS(p),$$

where dS represents a surface element on S^{m-1}.

2. NATIVE SPACES

In this section we introduce certain classes of zonal kernels on $S^{m-1} \times S^{m-1}$ and their associated native spaces. These kernels will form the foundation for the generalized Hermite interpolation discussed in the subsequent section.

Let κ be a zonal kernel with shape function $\psi_\kappa : [-1, 1] \to \mathbb{R}$. All kernels in this paper will be such that their shape function ψ_κ has a Legendre expansion of the form

$$\psi_\kappa(t) = \sum_{k=0}^{\infty} b_k(\psi_\kappa)\, P_k(t), \qquad t \in [-1, 1],$$

where the Legendre coefficients $b_k(\psi_\kappa)$ satisfy $b_k(\psi_\kappa) \geqslant 0$ and

$$\sum_{k=0}^{\infty} b_k(\psi_\kappa) < \infty. \tag{5}$$

By (4), these conditions guarantee that the above Legendre expansion converges uniformly and hence that κ is continuous. Another consequence is that κ is *positive definite*, which means that the matrix $(\kappa(p, q))_{p, q \in Q}$ is positive semi-definite for every finite collection of points $Q \subset S^{m-1}$. If in fact the coefficients $b_k(\psi_\kappa)$ are all positive, then the above-mentioned matrix is positive definite, in which case κ is called *strictly* positive definite. This is an important property since it implies, among other things, that the Lagrange interpolation problem

$$\sum_{q \in Q} c_q \kappa(p, q) = d_p, \qquad p \in Q,$$

is uniquely solvable for the coefficients c_q, for an arbitrary set Q of distinct points and arbitrary data values $d_p \in \mathbb{R}$. Positive definite functions on the sphere have received a lot of attention recently. For more details, see the survey [6] and the references therein.

A series expansion for $\kappa(p, q)$ can be obtained via the Addition Theorem (3),

$$\kappa(p, q) = \psi_\kappa(p \cdot q) = \sum_{k=0}^{\infty} \frac{\omega b_k(\psi_\kappa)}{N(k)} \sum_{l=1}^{N(k)} Y_{kl}(p)\, Y_{kl}(q), \tag{6}$$

or

$$\kappa(p, q) = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} Y_{kl}(p)\, Y_{kl}(q), \tag{7}$$

where

$$a_k(\kappa) := \frac{\omega b_k(\psi_\kappa)}{N(k)} \geqslant 0. \tag{8}$$

For convenience and conciseness we continue to use both notations $a_k(\kappa)$ and $b_k(\psi_\kappa)$, even though by (8), one of the two symbols could be removed.

There is a useful space associated with the kernel κ.

DEFINITION 1. The *native space* $\mathscr{H}_\kappa$ associated with the kernel κ is the space of all functions $u \in L_2$, for which $\hat{u}_{kl} = 0$ whenever $a_k(\kappa) = 0$, and such that

$$\|u\|_\kappa := \Bigg(\sum_{\substack{k=0 \\ a_k(\kappa) \neq 0}}^{\infty} \frac{1}{a_k(\kappa)} \sum_{l=1}^{N(k)} \hat{u}_{kl}^2 \Bigg)^{\frac{1}{2}} < \infty.$$

The Sobolev spaces H^s on the sphere (see Section 6) are a special instance of the native spaces, obtained by setting $a_k(\kappa) = (1 + \lambda_k)^{-s}$, $s \in \mathbb{R}$. The space $\mathscr{H}_\kappa$ is a Hilbert space, with associated inner product

$$\langle u, v \rangle_\kappa = \sum_{\substack{k=0 \\ a_k(\kappa) \neq 0}}^{\infty} \frac{1}{a_k(\kappa)} \sum_{l=1}^{N(k)} \hat{u}_{kl}\, \hat{v}_{kl}.$$

The following imbedding theorem implies that $\mathscr{H}_\kappa$ is a reproducing kernel Hilbert space.

PROPOSITION 2. *The Hilbert space $\mathscr{H}_\kappa$ is continuously imbedded in $C(S^{m-1})$.*

Proof. We need to prove that

$$\sup_{p \in S^{m-1}} |u(p)| \leqslant C \, \|u\|_\kappa, \qquad u \in \mathscr{H}_\kappa,$$

for some constant C that is independent of u. For every $p \in S^{m-1}$, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned} |u(p)| &\leqslant \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} |\hat{u}_{kl} Y_{kl}(p)| \\ &\leqslant \Bigg(\sum_{\substack{k=0 \\ a_k(\kappa) \neq 0}}^{\infty} \frac{1}{a_k(\kappa)} \sum_{l=1}^{N(k)} \hat{u}_{kl}^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} Y_{kl}^2(p) \Bigg)^{\frac{1}{2}} = C \, \|u\|_\kappa, \end{aligned}$$

with $C := (\omega^{-1} \sum_{k=0}^{\infty} a_k(\kappa)\, N(k))^{1/2} = (\sum_{k=0}^{\infty} b_k(\psi_\kappa))^{1/2} < \infty$. Similarly, it is possible to see that for $p, q \in S^{m-1}$ and $u \in \mathscr{H}_\kappa$,

$$|u(p) - u(q)| \leqslant \|u\|_\kappa \, (|\kappa(p, p) - \kappa(p, q)| + |\kappa(q, q) - \kappa(q, p)|)^{\frac{1}{2}},$$

which implies, by the continuity of κ, that u is continuous. ∎

The dual of the space $\mathscr{H}_\kappa$ will be denoted as $\mathscr{H}_\kappa^*$, which is also a Hilbert space with a corresponding norm $\|\cdot\|_{\kappa^*}$. It is not difficult to show that each $\lambda \in \mathscr{H}_\kappa^*$ can be associated with a real sequence $(\hat{\lambda}_{kl})$ such that $\hat{\lambda}_{kl} = 0$, $l = 1, \ldots, N(k)$, for k such that $a_k(\kappa) = 0$, and

$$\|\lambda\|_{\kappa^*} = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} \hat{\lambda}_{kl}^2 < \infty.$$

As a consequence, we have for $u \in \mathscr{H}_\kappa$ and $\lambda \in \mathscr{H}_\kappa^*$,

$$|\lambda u| = \left| \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} \hat{u}_{kl} \hat{\lambda}_{kl} \right| \leqslant \|\lambda\|_{\kappa^*} \, \|u\|_\kappa.$$

Native spaces were introduced by Madych and Nelson in [19, 20] to study the approximation properties of radial basis functions in $\mathbb{R}^m$. The spherical analog of native spaces has appeared in several papers [1, 14, 17]. In some sense, they can be viewed as a "discretized" version of the native spaces in $\mathbb{R}^m$ with the Fourier series in place of the standard Fourier transform.

A few remarks on the above definition of native spaces are in order. The definition simplifies if we assume that all coefficients $a_k(\kappa)$ are positive or, equivalently, that κ is strictly positive definite in the stronger distributional sense defined in [2]. In fact, this assumption is made in most of the previously quoted papers. If some of the $a_k(\kappa)$ are zero, then $\mathscr{H}_\kappa$ becomes degenerate in that it lacks spherical harmonics of certain frequencies. In this case $\mathscr{H}_\kappa$ is not dense in L_2. While in some settings this may be unacceptable, in the context of solving differential equations, degenerate native spaces arise naturally when dealing with operators, such as Δ^*, that annihilate a certain number of spherical harmonics.

It will be instructive to briefly motivate the definition of $\mathscr{H}_\kappa$. Frequently, the kernel κ is used to define functions of the form

$$\sum_{\mu \in \Lambda} c_\mu \mu^q \kappa(\cdot, q), \qquad c_\mu \in \mathbb{R}, \tag{9}$$

where Λ is a finite set of functionals. For example, $\mu \in \Lambda$ can be an evaluation functional, namely $\mu^q \kappa(\cdot, q) = \kappa(\cdot, q)$, $q \in S^{m-1}$, or a point-evaluation of a pseudodifferential operator, considered later in the paper. A question of fundamental importance is whether one can characterize functions that can be approximated, in a prescribed sense, by linear combinations of the form (9). It turns out that such functions are necessarily elements of $\mathscr{H}_\kappa$ if we let Λ be a subset of the dual space $\mathscr{H}_\kappa^*$. To see this, first note that a linear combination of the form (9) equals $\lambda^q \kappa(\cdot, q)$, where $\lambda := \sum_{\mu \in \Lambda} c_\mu \mu$ is also an element of $\mathcal{H}_\kappa^*$. We now claim that $\lambda^q\kappa(\cdot, q)$ is well defined and belongs to $\mathcal{H}_\kappa$. To do this, we must verify that $\kappa(\cdot, q) \in \mathcal{H}_\kappa$, for each fixed q. This follows from

$$\|\kappa(\cdot, q)\|_\kappa^2 = \kappa(q, q),$$

which is finite by the continuity of κ. To see that $\lambda^q\kappa(\cdot, q) \in \mathcal{H}_\kappa$, observe that

$$\|\lambda^q\kappa(\cdot, q)\|_\kappa^2 = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} (\lambda Y_{kl})^2 = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} \hat{\lambda}_{kl}^2 = \|\lambda\|_{\kappa^*}^2 < \infty$$

since $\lambda \in \mathcal{H}_\kappa^*$. In fact, the equality $\|\lambda^q\kappa(\cdot, q)\|_\kappa = \|\lambda\|_{\kappa^*}$ means that the space of linear combinations of the form (9), where Λ ranges over finite subsets of $\mathcal{H}_\kappa^*$, is identical to the space $\mathcal{H}_\kappa$. In applications we usually replace $\mathcal{H}_\kappa^*$ with a subspace of functionals of particular type (e.g., point-evaluation functionals) that is total for $\mathcal{H}_\kappa$. As a result, the space of functions of the form (9) will be dense in $\mathcal{H}_\kappa$. In other words, if a function u is to be approximated arbitrarily well by linear combinations of the form (9), then u has to be an element of $\mathcal{H}_\kappa$.

3. GENERALIZED HERMITE INTERPOLATION

Before considering differential equations on spheres, we discuss an abstract framework for generalized Hermite interpolation. For a finite collection of functionals $\Lambda \subset \mathcal{H}_\kappa^*$, let

$$S_{\Lambda,\kappa} := \operatorname{span}\{\lambda^q\kappa(\cdot, q) : \lambda \in \Lambda\}.$$

We are interested in the following

Problem 3. Let $u \in \mathcal{H}_\kappa$. Find an element $s_{u,\Lambda,\kappa} \in S_{\Lambda,\kappa}$ such that

$$\lambda s_{u,\Lambda,\kappa} = \lambda u, \qquad \lambda \in \Lambda. \tag{10}$$

This problem is not well posed unless the expression $\lambda s_{u,\Lambda,\kappa}$ is meaningful, which means that $\lambda^p\lambda^q\kappa(p, q)$ must be well defined for all $\lambda \in \Lambda$. However, as noted in the previous section, $\lambda^q\kappa(\cdot, q)$ is an element of $\mathcal{H}_\kappa$ and thus $\lambda^p\lambda^q\kappa(p, q)$ is indeed well defined. Moreover, $S_{\Lambda,\kappa}$ is a linear subspace of $\mathcal{H}_\kappa$ and therefore conditions (10) make sense.

Later, we shall need a formula for $\mu^p\nu^q\kappa(p, q)$, where $\mu, \nu \in \mathcal{H}_\kappa^*$. The expansion (7) for κ yields

$$\nu^q\kappa(\cdot, q) = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} Y_{kl}(\cdot)(\nu Y_{kl}),$$

where the equality holds in $\mathscr{H}_\kappa$. Note that the interchange of ν and the summation is justified by the fact that $\nu \in \mathscr{H}_\kappa^*$ and that for any fixed q, we have $\kappa(\cdot, q) \in \mathscr{H}_\kappa$. An immediate consequence of this and the fact that $\nu^q \kappa(\cdot, q) \in \mathscr{H}_\kappa$ is that for all $\mu, \nu \in \mathscr{H}_\kappa^*$,

$$\mu^p \nu^q \kappa(p, q) = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} (\mu Y_{kl})(\nu Y_{kl}). \tag{11}$$

Next we show that Problem 3 has a unique solution. Moreover, this solution is the best approximation to u from $S_{\Lambda, \kappa}$ in the topology of the native space $\mathscr{H}_\kappa$.

PROPOSITION 4. *Let $u \in \mathscr{H}_\kappa$ and let $\Lambda \subset \mathscr{H}_\kappa^*$ be a finite set. Then Problem 3 has a unique solution $s_{u, \Lambda, \kappa}$, which is such that*

$$\|u - v\|_\kappa^2 = \|u - s_{u, \Lambda, \kappa}\|_\kappa^2 + \|v - s_{u, \Lambda, \kappa}\|_\kappa^2,$$

for every $v \in S_{\Lambda, \kappa}$. In particular, $\|u - s_{u, \Lambda, \kappa}\|_\kappa = \inf_{v \in S_{\Lambda, \kappa}} \|u - v\|_\kappa$.

Proof. We only prove the existence and uniqueness of the function $s_{u, \Lambda, \kappa}$. The remaining part of the proposition is a standard variational result for Hilbert spaces, proved in a similar framework in [17].

Without loss of generality we can assume that the functionals in Λ are linearly independent. For it is easily seen that if this is not the case, then we can discard all redundant functionals in Λ without changing the space $S_{\Lambda, \kappa}$ and without altering the nature of conditions (10). With this assumption, Problem 3 gives rise to an invertible linear system of equations. To prove this, consider the matrix corresponding to the linear system (10). We show that this matrix is positive definite. Using (11), the quadratic form associated with the matrix is

$$\sum_{\mu, \nu \in \Lambda} c_\mu c_\nu \mu^p \nu^q \kappa(p, q) = \lambda^p \lambda^q \kappa(p, q) = \sum_{k=0}^{\infty} a_k(\kappa) \sum_{l=1}^{N(k)} (\lambda Y_{kl})^2,$$

where $\lambda := \sum_{\mu \in \Lambda} c_\mu \mu$, $c_\mu \in \mathbb{R}$. Since the coefficients $a_k(\kappa)$ are nonnegative, the quadratic form vanishes if and only if λ annihilates the spherical harmonics Y_{kl}, $l = 1, ..., N(k)$, corresponding to $a_k(\kappa) \neq 0$. Clearly the linear span of these spherical harmonics is dense in $\mathscr{H}_\kappa$ and hence $\lambda = 0$. By the linear independence of the functionals Λ (on $\mathscr{H}_\kappa$), the coefficients c_μ are identically zero. Consequently, the linear system of equations is uniquely solvable. ∎

To derive error estimates for the collocation method, we start with a version of a classical inequality [13]. This inequality uses the notion of a power function $P_{\kappa, \Lambda}(\mu)$ of a linear functional $\mu \in \mathscr{H}_{\kappa}^{*}$, associated with a zonal kernel κ and a finite set of linear functionals Λ. It is defined as

$$P_{\kappa, \Lambda}(\mu) := \inf_{\lambda \in \operatorname{span} \Lambda} \|\mu - \lambda\|_{\kappa^*}.$$

LEMMA 5. *Suppose that $v \in \mathscr{H}_{\kappa}$ is such that $\lambda v = 0$, for all $\lambda \in \Lambda$, and let $\mu \in \mathscr{H}_{\kappa}^{*}$. Then*

$$|\mu v| \leqslant P_{\kappa, \Lambda}(\mu) \, \|v\|_{\kappa}.$$

Proof. For any $\mu \in \mathscr{H}_{\kappa}^{*}$ we have by definition

$$\|\mu\|_{\kappa^*} = \sup_{\substack{v \in \mathscr{H}_{\kappa} \\ v \neq 0}} \frac{|\mu v|}{\|v\|_{\kappa}}.$$

If $\lambda \in \operatorname{span} \Lambda$, then $|\mu v| = |(\mu - \lambda)\, v| \leqslant \|\mu - \lambda\|_{\kappa^*} \|v\|_{\kappa}$. Taking the infimum over all such λ completes the proof. ∎

Since the solution $s_{u, \Lambda, \kappa}$ of Problem 3 satisfies the collocation conditions (10), a corollary of the previous lemma is

$$|\mu(u - s_{u, \Lambda, \kappa})| \leqslant P_{\kappa, \Lambda}(\mu) \, \|u - s_{u, \Lambda, \kappa}\|_{\kappa}, \qquad \mu \in \mathscr{H}_{\kappa}^{*}. \tag{12}$$

Thus this inequality isolates the problem of approximating the linear functional μ by the finite set of linear functionals Λ, from the problem of how well the solution u can be approximated by a function in $S_{\Lambda, \kappa}$.

The majority of study of the approximation properties of zonal kernels on S^{m-1} has concentrated on just one of the factors in inequality (12)—the power function. In this paper we will investigate both factors, and under favorable conditions we will be able to collect approximation order from both of them to effectively double the approximation order. This idea first appears in [23] for $\mathbb{R}^m$.

The framework for generalized Hermite interpolation that we presented in this section is not the only approach. Another alternative is to use Sobolev spaces as in [2].

4. THE COLLOCATION PROBLEM FOR PSEUDODIFFERENTIAL EQUATIONS

Let $(L^\wedge(k))_{k\geqslant 0}$ be a polynomially bounded sequence of real numbers and suppose that L is an operator that assigns to any $u \in L_2$ the expression

$$Lu = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} L^\wedge(k)\, \hat{u}_{kl} Y_{kl}. \tag{13}$$

In particular,

$$LY_{kl} = L^\wedge(k)\, Y_{kl},$$

for all $l = 1, ..., N(k)$, $k \geqslant 0$. Note that Lu is an element of $\mathscr{D}'$, the space of distributions on S^{m-1}, in the sense that $\langle Lu, \varphi\rangle := \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} L^\wedge(k)\, \hat{u}_{kl} \hat{\varphi}_{kl}$, $\varphi \in C^\infty(S^{m-1})$, defines a continuous linear functional on $C^\infty(S^{m-1})$. The operator L is called a *pseudodifferential operator* and the sequence $(L^\wedge(k))_{k\geqslant 0}$ is referred to as the *spherical symbol of* L. A pseudodifferential operator L is said to be of *order* t if there exist positive constants C_1 and C_2 such that

$$C_1(1+\lambda_k)^{t/2} \leqslant |L^\wedge(k)| \leqslant C_2(1+\lambda_k)^{t/2}, \tag{14}$$

for all k such that $L^\wedge(k) \neq 0$. This definition of pseudodifferential operators appeared essentially in [10] (see also [25]) and was motivated by the classical theory of pseudodifferential operators in $\mathbb{R}^m$ [3]. Examples are the Laplace–Beltrami operator Δ^*, a pseudodifferential operator of order two, for which $L^\wedge(k) = -\lambda_k$, and the operator

$$L = -\Delta^* + \left(\frac{m-2}{2}\right)^2,$$

with spherical symbol

$$L^\wedge(k) = \left(k + \frac{m-2}{2}\right)^2.$$

Moreover, boundary value problems of physical geodesy in $\mathbb{R}^3$ can be typically reformulated as pseudodifferential equations on S^2 [25]. Many other examples are given in [10].

To investigate the collocation problem described in Section 1, for a pseudodifferential operator L and a kernel κ, let

$$\kappa_L(p, q) := L^p L^q \kappa(p, q).$$

As mentioned earlier, this kernel will play an important role in our analysis. To be able to apply the results of the previous section to κ_L, we need to verify that under appropriate conditions this kernel is well defined and that its properties are similar to the properties of the original kernel κ.

LEMMA 6. *Let κ be a zonal kernel satisfying* (5) *and*

$$\sum_{k=0}^{\infty} L^\wedge(k)^2\, b_k(\psi_\kappa) < \infty. \tag{15}$$

Then κ_L is a continuous zonal kernel with coefficients $a_k(\kappa_L) = a_k(\kappa)\, L^\wedge(k)^2 \geqslant 0$.

Proof. We check that $\lambda := \delta_p \circ L \in \mathscr{H}_\kappa^*$, $p \in S^{m-1}$, where δ_p denotes the functional, defined for a function g as $\delta_p g := g(p)$. To see this, we note that if $u \in \mathscr{H}_\kappa$, then by the Cauchy–Schwarz inequality,

$$\begin{aligned} |Lu(p)| &\leqslant \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} |L^\wedge(k)\, \hat{u}_{kl} Y_{kl}(p)| \\ &\leqslant \left(\sum_{k=0}^{\infty} a_k(\kappa)\, L^\wedge(k)^2 \sum_{l=1}^{N(k)} Y_{kl}^2(p) \right)^{\frac{1}{2}} \left(\sum_{\substack{k=0 \\ a_k(\kappa) \neq 0}}^{\infty} \frac{1}{a_k(\kappa)} \sum_{l=1}^{N(k)} \hat{u}_{kl}^2 \right)^{\frac{1}{2}} \\ &= C\, \|u\|_\kappa, \end{aligned}$$

with $C := (\sum_{k=0}^{\infty} L^\wedge(k)^2\, b_k(\psi_\kappa))^{\frac{1}{2}} < \infty$. Hence, by the remarks after Problem 3 we know that $\lambda^q \kappa(\cdot, q) = L^q \kappa(\cdot, q) \in \mathscr{H}_\kappa$ and that $\lambda^p \lambda^q \kappa(p, q) = L^p L^q \kappa(p, q)$ is well defined. Moreover, by (13) the new kernel $\kappa_L(p, q)$ takes the form

$$\kappa_L(p, q) = \sum_{k=0}^{\infty} a_k(\kappa)\, L^\wedge(k)^2 \sum_{l=1}^{N(k)} Y_{kl}(p)\, Y_{kl}(q).$$

This shows that κ_L is zonal and that $a_k(\kappa_L) = a_k(\kappa)\, L^\wedge(k)^2 \geqslant 0$. The assumption (15), together with (4), implies that κ_L is continuous. ∎

It follows from the previous lemma that there is a well-defined native space $\mathscr{H}_{\kappa_L}$ associated with κ_L. It is not difficult to see that L maps the space $\mathscr{H}_\kappa$ onto $\mathscr{H}_{\kappa_L}$ and hence the pseudodifferential equation

$$Lu = f, \qquad f \in \mathscr{H}_{\kappa_L},$$

is solvable in $\mathscr{H}_\kappa$. However, the solution may not be unique if L is not invertible. In Section 6 we will impose additional conditions on u that will determine a unique solution.

We are now ready to formulate the collocation problem for solving pseudodifferential equations more precisely. This will be a special case of Problem 3, in which we restrict ourselves to a special class of functionals Λ, namely point-evaluations of the operator L.

Problem 7. Let Q be a finite set of distinct collocation points in S^{m-1} and let

$$\Lambda = \{\delta_q \circ L\colon \mathscr{H}_\kappa \to \mathbb{R},\ q \in Q\}.$$

Also, let $f \in \mathscr{H}_{\kappa_L}$ and $u \in \mathscr{H}_\kappa$ be such that

$$Lu = f. \tag{16}$$

Find a function $s_{u,\Lambda,\kappa} \in S_{\Lambda,\kappa}$ satisfying the collocation conditions

$$\lambda s_{u,\Lambda,\kappa} = \lambda u, \qquad \lambda \in \Lambda,$$

or, equivalently,

$$L s_{u,\Lambda,\kappa}(q) = f(q), \qquad q \in Q.$$

5. ANALYSIS OF APPROXIMATION ORDER

Proposition 4, applied to κ_L instead of κ, combined with Lemma 6, implies that Problem 7 has a unique solution $s_{u,\Lambda,\kappa}$. In this and the next section we will be interested in how well this function approximates the exact solution of the pseudodifferential Eq. (16). First we will establish bounds on the expression $Lu - Ls_{u,\Lambda,\kappa} = f - Ls_{u,\Lambda,\kappa}$. Estimates for the error $u - s_{u,\Lambda,\kappa}$ will be obtained in the next section. The estimates will be given explicitly in terms of a mesh norm h that measures the density of the collocation points Q, defined as

$$h := \sup_{p \in S^{m-1}} \min_{q \in Q} d(p,q),$$

where $d(p,q)$ is the geodesic distance of $p, q \in S^{m-1}$.

We will adapt a technique used in [8] for $\mathbb{R}^m$ to the sphere S^{m-1}. This technique reduces the problem of finding convergence order results for Problem 7 to establishing the order of convergence for a particular Lagrange interpolation problem. Consider the pointwise error

$$|L(u - s_{u,\Lambda,\kappa})(p)|$$

at an arbitrary point $p \in S^{m-1}$. As we know, $s_{u, \Lambda, \kappa}$ is the best approximation (in the native space $\mathscr{H}_\kappa$) to u from the space $S_{\Lambda, \kappa}$. We now claim that $Ls_{u, \Lambda, \kappa}$ is the best approximation in $\mathscr{H}_{\kappa_L}$ to f from S_{δ_Q, κ_L}, where $\delta_Q := \{\delta_q : q \in Q\}$. Note that $\delta_Q \subset \mathscr{H}^*_{\kappa_L}$, which follows from the fact that by (15) and by Proposition 2 (applied to κ_L instead of κ), $\mathscr{H}_{\kappa_L}$ is continuously imbedded in $C(S^{m-1})$.

LEMMA 8. $Ls_{u, \Lambda, \kappa} = s_{f, \delta_Q, \kappa_L}$.

Proof. The proof is done in a similar way to the case of $\mathbb{R}^m$ (see [8, Theorem 2.8]). By definition, $s_{u, \Lambda, \kappa}$ is such that $\delta_q(Ls_{u, \Lambda, \kappa}) = \delta_q(Lu) = f(q)$, $q \in Q$. Moreover, $Ls_{u, \Lambda, \kappa} \in S_{\delta_Q, \kappa_L}$. On the other hand, $s_{f, \delta_Q, \kappa_L} \in S_{\delta_Q, \kappa_L}$ is the unique function satisfying $\delta_q(s_{f, \delta_Q, \kappa_L}) = f(q)$, $q \in Q$. Hence $Ls_{u, \Lambda, \kappa} = s_{f, \delta_Q, \kappa_L}$. ∎

Combining Lemma 8 with inequality (12) leads to

$$|L(u - s_{u, \Lambda, \kappa})(p)| = |\delta_p(f - s_{f, \delta_Q, \kappa_L})| \leqslant P_{\kappa_L, \delta_Q}(\delta_p) \, \|f - s_{f, \delta_Q, \kappa_L}\|_{\kappa_L}, \tag{17}$$

for every $p \in S^{m-1}$. To bound the right-hand side of (17), we first need an estimate for the power function $P_{\kappa_L, \delta_Q}(\delta_p)$. To do this, we modify an argument given in [14] for Lagrange interpolation on the sphere. The following result from [14, Proposition 2] will be crucial.

LEMMA 9. *Let $p \in S^{m-1}$ and let Q have mesh norm $h \leqslant 1/(2K)$, for some $K \in \mathbb{N}$. Then there exist numbers $c_q \in \mathbb{R}$, $q \in Q$, such that*

$$\left(\delta_p - \sum_{q \in Q} c_q \delta_q \right) Y_{kl} = 0,$$

for all $l = 1, ..., N(k)$, $k = 0, ..., K$, and such that

$$\sum_{q \in Q} |c_q| \leqslant 2.$$

It is now possible to estimate the power function $P_{\kappa_L, \delta_Q}(\delta_p)$. From now on we will assume that the integer K is always such that $h \leqslant 1/(2K)$ holds.

PROPOSITION 10. *Let κ_L be a zonal kernel with coefficients satisfying* (5) *and* (15). *Then*

$$P_{\kappa_L, \delta_Q}(\delta_p) \leqslant 3 \left(\sum_{k > K} b_k(\psi_{\kappa_L}) \right)^{\frac{1}{2}},$$

for all $p \in S^{m-1}$.

Proof. Let $p \in S^{m-1}$ be fixed. Also, let c_q, $q \in Q$, be the numbers from Lemma 9, and $c_p := -1$. We have

$$\begin{aligned}
P_{\kappa_L, \delta_Q}(\delta_p) &\leqslant \sup_{\substack{v \in \mathcal{H}_{\kappa_L} \\ \|v\|_{\kappa_L}=1}} \left|\left(\delta_p - \sum_{q \in Q} c_q \delta_q\right) v\right| \\
&\leqslant \sup_{\substack{v \in \mathcal{H}_{\kappa_L} \\ \|v\|_{\kappa_L}=1}} \sum_{q \in Q \cup \{p\}} |c_q| \left|\sum_{k>K} \sum_{l=1}^{N(k)} \hat{v}_{kl} Y_{kl}(q)\right| \\
&\leqslant \sup_{\substack{v \in \mathcal{H}_{\kappa_L} \\ \|v\|_{\kappa_L}=1}} \left(\sum_{q \in Q \cup \{p\}} |c_q|\right) \max_{q \in Q \cup \{p\}} \left|\sum_{k>K} \sum_{l=1}^{N(k)} \hat{v}_{kl} Y_{kl}(q)\right| \\
&\leqslant 3 \sup_{\substack{v \in \mathcal{H}_{\kappa_L} \\ \|v\|_{\kappa_L}=1}} \max_{q \in Q \cup \{p\}} \left|\sum_{k>K} \sum_{l=1}^{N(k)} \hat{v}_{kl} Y_{kl}(q)\right|.
\end{aligned}$$

To evaluate the maximum value above, we apply Cauchy–Schwarz inequality to obtain

$$\begin{aligned}
\left(\sum_{k>K} \sum_{l=1}^{N(k)} \hat{v}_{kl} Y_{kl}(q)\right)^2 &\leqslant \left(\sum_{\substack{k>K \\ a_k(\kappa_L) \neq 0}} \sum_{l=1}^{N(k)} \frac{\hat{v}_{kl}^2}{a_k(\kappa_L)}\right)\left(\sum_{k>K} a_k(\kappa_L) \sum_{l=1}^{N(k)} Y_{kl}^2(q)\right) \\
&\leqslant \|v\|_{\kappa_L}^2 \sum_{k>K} \frac{a_k(\kappa_L)\, N(k)}{\omega} = \|v\|_{\kappa_L}^2 \sum_{k>K} b_k(\psi_{\kappa_L}).
\end{aligned}$$

Combining the above inequalities concludes the proof. ∎

Remark 11. A similar result to Proposition 10, albeit for a slightly larger class of kernels, was proved in [14, Theorem 2], namely that, in our notation,

$$P_{\kappa_L, \delta_Q}(\delta_p) \leqslant 5(\#Q+1)\left(\sum_{k>K} b_k(\psi_{\kappa_L})\right)^{\frac{1}{2}},$$

where $\#Q$ is the cardinality of Q. Thus we have improved this result (for the class of kernels considered here) by removing the factor $\#Q+1$ from the bound. Moreover, we have done away with the assumption that all $a_k(\kappa_L)$ must be positive.

We now turn our attention to the second factor in (17), $\|f - s_{f, \delta_Q, \kappa_L}\|_{\kappa_L}$, which is a classical error for Lagrange interpolation. By Proposition 4, a crude bound for this error is

$$\|f - s_{f, \delta_Q, \kappa_L}\|_{\kappa_L} \leqslant \|f\|_{\kappa_L}.$$

However, if we assume additional smoothness on f, then we can employ an idea from [23] to get a tighter bound for $\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}$. For a kernel κ with expansion (7) we can define the space

$$\mathscr{H}_{\kappa*\kappa} := \left\{ v \in \mathscr{H}_\kappa : \|v\|_{\kappa*\kappa} := \left(\sum_{\substack{k=0 \\ a_k(\kappa)\neq 0}}^{\infty} \frac{1}{a_k^2(\kappa)} \sum_{l=1}^{N(k)} \hat{v}_{kl}^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

The notation $\mathscr{H}_{\kappa*\kappa}$, which is suggestive of convolution, is used because it can be shown that $\mathscr{H}_{\kappa*\kappa}$ is the native space of the kernel $\kappa*\kappa$ defined by

$$(\kappa*\kappa)(p,q) := \int_{S^{m-1}} \kappa(p,r)\,\kappa(r,q)\,dS(r), \qquad p,q \in S^{m-1}.$$

Proposition 12. *For all* $f \in \mathscr{H}_{\kappa_L*\kappa_L}$,

$$\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L} \leqslant 3\sqrt{\omega}\,\|f\|_{\kappa_L*\kappa_L} \left(\sum_{k>K} b_k(\psi_{\kappa_L}) \right)^{\frac{1}{2}}.$$

Proof. Since s_{f,δ_Q,κ_L} is the best approximation to f from $\mathscr{H}_{\kappa_L}$, we have

$$\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}^2 = \langle f-s_{f,\delta_Q,\kappa_L}, f-s_{f,\delta_Q,\kappa_L}\rangle_{\kappa_L} = \langle f, f-s_{f,\delta_Q,\kappa_L}\rangle_{\kappa_L}.$$

By the definition of $\mathscr{H}_{\kappa_L}$ and the Cauchy–Schwarz inequality,

$$\begin{aligned}
\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}^2 &= \langle f, f-s_{f,\delta_Q,\kappa_L}\rangle_{\kappa_L} \\
&= \sum_{\substack{k=0 \\ a_k(\kappa_L)\neq 0}}^{\infty} \sum_{l=1}^{N(k)} \frac{\hat{f}_{kl}(\hat{f}_{kl}-(\hat{s}_{f,\delta_Q,\kappa_L})_{kl})}{a_k(\kappa_L)} \\
&\leqslant \left(\sum_{\substack{k=0 \\ a_k(\kappa_L)\neq 0}}^{\infty} \sum_{l=1}^{N(k)} \frac{\hat{f}_{kl}^2}{a_k^2(\kappa_L)} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} (\hat{f}_{kl}-(\hat{s}_{f,\delta_Q,\kappa_L})_{kl})^2 \right)^{\frac{1}{2}} \\
&= \|f\|_{\kappa_L*\kappa_L}\,\|f-s_{f,\delta_Q,\kappa_L}\|_{L_2}.
\end{aligned} \tag{18}$$

By Lemma 5, with $\mu=\delta_p$ and $\Lambda=\delta_Q$,

$$|f(p)-s_{f,\delta_Q,\kappa_L}(p)| \leqslant P_{\kappa_L,\delta_Q}(\delta_p)\,\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}.$$

Hence, integration with respect to p, along with Proposition 10, yields

$$\|f-s_{f,\delta_Q,\kappa_L}\|_{L_2} \leqslant 3\sqrt{\omega}\left(\sum_{k>K} b_k(\psi_{\kappa_L}) \right)^{\frac{1}{2}} \|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}. \tag{19}$$

Combining inequalities (18) and (19), we have

$$\|f-s_{f,\delta_Q,\kappa_L}\|^2_{\kappa_L} \leqslant 3\sqrt{\omega}\,\|f\|_{\kappa_L*\kappa_L}\Big(\sum_{k>K} b_k(\psi_{\kappa_L})\Big)^{\frac{1}{2}}\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}.$$

Canceling a factor of $\|f-s_{f,\delta_Q,\kappa_L}\|_{\kappa_L}$ gives the desired result. ∎

Proposition 10, together with Proposition 12, yields the main result of this section.

THEOREM 13. *In the notation of Problem 7,*

$$|Lu(p)-Ls_{u,\Lambda,\kappa}(p)| \leqslant 3\,\|f\|_{\kappa_L}\Big(\sum_{k>K} L^\wedge(k)^2\, b_k(\psi_\kappa)\Big)^{\frac{1}{2}},$$

for all $p \in S^{m-1}$. If in addition $f \in \mathscr{H}_{\kappa_L*\kappa_L}$, then

$$|Lu(p)-Ls_{u,\Lambda,\kappa}(p)| \leqslant 3\sqrt{\omega}\,\|f\|_{\kappa_L*\kappa_L}\sum_{k>K} L^\wedge(k)^2\, b_k(\psi_\kappa).$$

The above estimates depend implicitly on the mesh norm h since K is a function of h. In the next section we make this connection explicit by assuming a certain decay on the coefficients $b_k(\psi_\kappa)$, which is related to the smoothness of κ, and on the decay of the spherical symbol $L^\wedge(k)$, which determines the order of the pseudodifferential operator L.

6. EXPLICIT ERROR BOUNDS

In the previous section we derived an error estimate for the approximation of Lu by $Ls_{u,\Lambda,\kappa}$. However, for practical purposes it is desirable to have an approximation not just to Lu, but to the solution u. Up to this point u can be any function in $\mathscr{H}_\kappa$ that satisfies $Lu = f$. Unfortunately, if the numbers $L^\wedge(k)$ are not all nonzero, then such a function may not be unique. In that case the various solutions may differ at the frequencies k for which $L^\wedge(k) = 0$. In this section we will impose additional conditions on the function u that will guarantee its uniqueness. Then we prove an error bound on the distance between u and its approximation s_u. This distance will be measured in a Sobolev norm, which is a natural norm in the setting of solving pseudodifferential equations.

To address the uniqueness of the solution of the equation $Lu = f$, let us define the index set

$$K_0(L) := \{k \colon L^\wedge(k) = 0\}$$

and the associated set of spherical harmonics

$$\Pi_L := \text{span}\{Y_{kl}: l = 1, \ldots, N(k), k \in K_0(L)\}.$$

Let Γ be a set of continuous linear functionals on $\mathcal{H}_\kappa$ that is unisolvent relative to Π_L. This means that for every set of data values $\{d_\gamma \in \mathbb{R} : \gamma \in \Gamma\}$, there exists a unique element $u_0 \in \Pi_L$ such that $\gamma u_0 = d_\gamma$, $\gamma \in \Gamma$. Since point-evaluations are continuous on $\mathcal{H}_\kappa$, a possible choice of Γ is

$$\Gamma = \{\delta_r : r \in R\},$$

where R is a set of distinct points in S^{m-1} for which Γ is unisolvent. In the sequel we restrict ourselves to pseudodifferential operators for which $K_0(L)$, and hence Γ, is finite (see Remark 17). This is a reasonable assumption since $K_0(L)$ is finite for the majority of useful operators. For example, for Δ^* we have $K_0(L) = \{0\}$.

Let us now return to the question of solvability of the equation $Lu = f$.

LEMMA 14. *Let L be a pseudodifferential operator, and let $f \in \mathcal{H}_{\kappa_L}$. Moreover, let $d_\gamma \in \mathbb{R}$, $\gamma \in \Gamma$, be given. Then there exists a unique solution $u \in \mathcal{H}_\kappa$ of the equation*

$$Lu = f, \tag{20}$$

such that

$$\gamma u = d_\gamma, \qquad \gamma \in \Gamma. \tag{21}$$

Proof. We can decompose the native space $\mathcal{H}_\kappa$ as

$$\mathcal{H}_\kappa = \Pi_L \oplus \Pi_L^c,$$

where $\Pi_L^c := \{v \in \mathcal{H}_\kappa : \hat{v}_{kl} = 0\,, k \in K_0(L)\}$ is the complementary space to Π_L. Thus every solution $u \in \mathcal{H}_\kappa$ of the system (20)–(21) can be written in the form

$$u = u_0 + \bar{u},$$

where $u_0 \in \Pi_L$ and $\bar{u} \in \Pi_L^c$. Observe that (20) is equivalent to $L\bar{u} = f$, which is uniquely solvable for $\bar{u} \in \Pi_L^c$ since $L: \Pi_L^c \to \mathcal{H}_{\kappa_L}$ is invertible. The inverse of L is determined by the symbol

$$(L^{-1})^\wedge(k) = \begin{cases} 1/L^\wedge(k), & k \notin K_0(L), \\ 0, & k \in K_0(L). \end{cases}$$

Equations (21) can now be written as $\gamma u_0 = d_\gamma - \gamma\bar{u}$, $\gamma \in \Gamma$, which can be uniquely solved (for u_0) by the unisolvency of Γ. ∎

We now recall some useful facts about spherical analogs of Sobolev spaces [16]. These are defined as

$$H^s := H^s(S^{m-1}) := \left\{ v \in \mathscr{D}' : \|v\|_s := \left(\sum_{k=0}^{\infty} (1+\lambda_k)^s \sum_{l=1}^{N(k)} \hat{v}_{kl}^2 \right)^{\frac{1}{2}} < \infty \right\}.$$

If $v \in H^s$ and $s > t$, then clearly $\|v\|_t \leqslant \|v\|_s$. A classical result about Sobolev spaces is the Sobolev Imbedding Theorem [11, p. 35], which asserts that whenever $s > n + (m-1)/2$, for some $n \geqslant 0$, then H^s is continuously imbedded in $C^n(S^{m-1})$.

We note that every pseudodifferential operator L of order t maps H^s to H^{s-t}, $s \in \mathbb{R}$. In addition, L has the following property:

PROPOSITION 15. *Let L be a pseudodifferential operator of order t, and let Γ be a unisolvent set of continuous linear functionals on H^s relative to Π_L. Then there exists a constant C_L such that for every $v \in H^s$ satisfying $\gamma v = 0$, $\gamma \in \Gamma$,*

$$\|v\|_s \leqslant C_L \, \|Lv\|_{s-t}.$$

Proof. Suppose that the assertion of the theorem is not true. Then there exists a sequence $(v^i) \in H^s$ such that $\gamma v^i = 0$, $\gamma \in \Gamma$, and

$$1 = \|v^i\|_s > i \, \|Lv^i\|_{s-t}.$$

This means that $\lim_{i \to \infty} \|Lv^i\|_{s-t} = 0$, or equivalently,

$$\lim_{i \to \infty} \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} (\hat{v}_{kl}^i)^2 \, L^\wedge(k)^2 \, (1+\lambda_k)^{s-t} = 0. \tag{22}$$

Let $v^i = v_0^i + \bar{v}^i$, where v_0^i is the Π_L-component of v^i. Clearly, $1 = \|v^i\|_s^2 = \|v_0^i\|_s^2 + \|\bar{v}^i\|_s^2$, and therefore

$$\|v_0^i\|_s \leqslant 1. \tag{23}$$

On the other hand, by (14) there exists a positive constant C_1 such that

$$\|v_0^i\|_s^2 = 1 - \|\bar{v}^i\|_s^2 \geqslant 1 - \frac{1}{C_1} \sum_{k \notin K_0(L)} \sum_{l=1}^{N(k)} (1+\lambda_k)^{s-t} \, L^\wedge(k)^2 \, (\hat{v}_{kl}^i)^2.$$

This, together with (22) and (23), yields

$$\lim_{i \to \infty} \|v_0^i\|_s = 1, \tag{24}$$

and $\lim_{i \to \infty} \|\bar{v}^i\|_s = 0$. It follows by the continuity of the linear functionals in Γ that $\lim_{i \to \infty} \gamma \bar{v}^i = 0$, $\gamma \in \Gamma$. Combining this with the fact that $\gamma v^i = 0$, leads to

$$\lim_{i \to \infty} \gamma v_0^i = 0, \qquad \gamma \in \Gamma. \tag{25}$$

The unisolvency of Γ implies that we can define a norm on Π_L by

$$\|w\|_\Gamma := \sum_{\gamma \in \Gamma} |\gamma w|, \qquad w \in \Pi_L.$$

Moreover, by the finite dimensionality of Π_L, this norm is equivalent to all other norms on this space. In particular, by (25), $\lim_{i \to \infty} \|v_0^i\|_\Gamma = 0$, whence

$$\lim_{i \to \infty} \|v_0^i\|_s = 0.$$

This is in direct contradiction to (24). ∎

With the above preparatory results, we now revisit the issue of solving pseudodifferential equations by collocation. Motivated by Lemma 14, consider the system of equations (20)–(21). The proof of this lemma suggests that we think of u as a sum of the form $u = u_0 + \bar{u}$, where u_0 is the Π_L-component of u. We then approximate u in two stages. In the first stage, an approximation $s_{\bar{u}, \Lambda, \kappa} \in S_{\Lambda, \kappa}$ of the function $\bar{u}$ is obtained by collocation. In the second stage, we approximate u_0 by a polynomial $s_0 \in \Pi_L$, determined by solving a Lagrange interpolation problem. Specifically, given the numbers d_γ, $\gamma \in \Gamma$, the approximation

$$s_u := s_0 + s_{\bar{u}, \Lambda, \kappa} \tag{26}$$

of u is obtained by solving the linear system

$$\begin{aligned} Ls_u(q) &= f(q), \qquad & q \in Q, \\ \gamma s_u &= d_\gamma, \qquad & \gamma \in \Gamma. \end{aligned} \tag{27}$$

Note that the two types of equations can be decoupled. Namely, since L annihilates s_0, the first set of equations is equivalent to

$$Ls_{\bar{u}, \Lambda, \kappa}(q) = f(q), \qquad q \in Q. \tag{28}$$

After solving for $s_{\bar{u}, \Lambda, \kappa}$, we can proceed with the second set of equations and solve for s_0. These can be written as

$$\gamma s_0 = d_\gamma - \gamma s_{\bar{u}, \Lambda, \kappa}, \qquad \gamma \in \Gamma.$$

Notice that in order to determine s_u it is not necessary to compute the decomposition $u = u_0 + \bar{u}$ explicitly.

We are now ready to state the result alluded to earlier asserting that the error of approximation of the collocation method can be bounded by an expression that depends explicitly on the mesh size h of the set of collocation points. The next theorem is the main result of this paper. It is a corollary of Theorem 13 and Proposition 15, and we state it just for the stronger case when $f \in \mathcal{H}_{\kappa_L * \kappa_L}$. For convenience and ease of reference, below we explicitly collect all the required assumptions.

THEOREM 16. *Let L be a pseudodifferential operator of order $t \in \mathbb{R}$, for which $K_0(L)$ is finite, let κ be a zonal kernel satisfying* (5) *and* (15), *and let $f \in \mathcal{H}_{\kappa_L * \kappa_L}$. Suppose that $u \in \mathcal{H}_\kappa$ is a solution of the equation $Lu = f$, subject to $\gamma u = d_\gamma$, $\gamma \in \Gamma$, where $\Gamma \subset (H^t)^*$ is a unisolvent set relative to Π_L, and $d_\gamma \in \mathbb{R}$. Then u is uniquely determined. Moreover, there exists a unique function s_u of the form* (26), *solving the collocation problem* (27). *The order of approximation of s_u to u is as follows:*

(a) *Suppose that there exists a constant C_κ such that $b_k(\psi_\kappa) \leqslant C_\kappa (1+k)^{-\alpha-1}$, for all $k \geqslant 0$ and some $\alpha > 2t$. Then there exists a C independent of h and f such that*

$$\|u - s_u\|_t \leqslant C h^{\alpha - 2t} \|f\|_{\kappa_L * \kappa_L}.$$

(b) *Suppose that there exists a constant C_κ such that $b_k(\psi_\kappa) \leqslant C_\kappa e^{-\beta k}$, for all $k \geqslant 0$ and some $\beta > 0$. Then there exists a C independent of h and f such that*

$$\|u - s_u\|_t \leqslant C h^{-2t} e^{-\beta/2h} \|f\|_{\kappa_L * \kappa_L}.$$

Proof. To be able to apply Lemma 14, we must verify that the assumption $\Gamma \subset (H^t)^*$ is sufficient to guarantee that $\Gamma \subset \mathcal{H}_\kappa^*$. From (14) and (15) we see that $\lim_{k \to \infty} (1+\lambda_k)^t \, b_k(\psi_\kappa) = 0$, and hence by (8), $(1+\lambda_k)^t \leqslant C_t / a_k(\kappa)$, $a_k(\kappa) \neq 0$, for some $C_t > 0$. Thus for every $v \in \mathcal{H}_\kappa$,

$$\|v\|_t^2 = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} \hat{v}_{kl}^2 (1+\lambda_k)^t \leqslant C_t \sum_{\substack{k=0 \\ a_k(\kappa) \neq 0}}^{\infty} \sum_{l=1}^{N(k)} \frac{\hat{v}_{kl}^2}{a_k(\kappa)} = C_t \|v\|_\kappa^2.$$

As a consequence, $\mathcal{H}_\kappa$ is continuously imbedded in H^t and hence $\Gamma \subset \mathcal{H}_\kappa^*$. By Lemma 14 and the fact that $f \in \mathcal{H}_{\kappa_L * \kappa_L} \subset \mathcal{H}_{\kappa_L}$, the function u is uniquely determined.

By Proposition 4, the function $s_{\bar{u}, \Lambda, \kappa} \in S_{\Lambda, \kappa}$ is uniquely determined by Eq. (28). Moreover, the term s_0 is found by solving the linear system

$$\gamma s_0 = d_\gamma - \gamma s_{\bar{u}, \Lambda, \kappa}, \qquad \gamma \in \Gamma,$$

which is invertible by the unisolvency of Γ. This proves that $s_u = s_0 + s_{\bar{u}, \Lambda, \kappa}$ exists and is unique.

To prove assertion (a), let K be the largest integer such that $h \leqslant 1/(2K)$ (cf. Lemma 9), so that

$$\frac{1}{K+1} \leqslant 2h \leqslant \frac{1}{K}. \tag{29}$$

From Theorem 13 we have for any $p \in S^{m-1}$,

$$\begin{aligned} |L\bar{u}(p) - Ls_{\bar{u}, \Lambda, \kappa}(p)| &\leqslant 3\sqrt{\omega}\, \|f\|_{\kappa_L * \kappa_L} \sum_{k>K} L\hat{}(k)^2\, b_k(\psi_\kappa) \\ &\leqslant 3\sqrt{\omega}\, C_2 C_\lambda C_\kappa\, \|f\|_{\kappa_L * \kappa_L} \sum_{k>K} (1+k)^{-\alpha-1+2t}, \end{aligned}$$

where C_λ satisfies $(1+\lambda_k)^t \leqslant C_\lambda (1+k)^{2t}$, $k \geqslant 0$. Bounding the sum of the series, we have

$$\begin{aligned} \sum_{k>K} (1+k)^{-\alpha-1+2t} &\leqslant \int_K^\infty (1+x)^{-\alpha-1+2t}\, dx \\ &\leqslant \frac{1}{(\alpha-2t)(K+1)^{\alpha-2t}} \leqslant \frac{(2h)^{\alpha-2t}}{(\alpha-2t)}, \end{aligned}$$

where the last line uses (29). Combining the above inequalities, we obtain

$$\|L\bar{u} - Ls_{\bar{u}, \Lambda, \kappa}\|_\infty \leqslant \frac{3\sqrt{\omega}\, C_2 C_\lambda C_\kappa 2^{\alpha-2t}}{(\alpha-2t)}\, h^{\alpha-2t}\, \|f\|_{\kappa_L * \kappa_L}.$$

To complete the proof, by Proposition 15, applied with $v = u - s_u \in \mathscr{H}_\kappa \subset H^t$ and $s = t$, there exists a positive constant C_L such that

$$\begin{aligned} \|u - s_u\|_t &\leqslant C_L\, \|Lu - Ls_u\|_0 \\ &= C_L\, \|L\bar{u} - Ls_{\bar{u}, \Lambda, \kappa}\|_{L_2} \\ &\leqslant C_L \sqrt{\omega}\, \|L\bar{u} - Ls_{\bar{u}, \Lambda, \kappa}\|_\infty. \end{aligned}$$

The claim of the theorem now follows. Part (b) can be proved along similar lines. ∎

It will be instructive to compare Theorem 16 to related work on Hermite interpolation [15], and on Lagrange interpolation [12, 14]. Some of these cited papers assume certain continuity of the shape function ψ_κ, rather than conditions on the decay of the $b_k(\psi_\kappa)$. To be able to make a comparison, let us link the decay of $b_k(\psi_\kappa)$ to the continuity properties of the shape function ψ_κ. If we assume that

$$b_k(\psi_\kappa) = O((k+1)^{-2n-1-\varepsilon}), \tag{30}$$

for some $\varepsilon > 0$, $n \geqslant 0$, then it follows from the well-known Bernstein inequality for trigonometric polynomials that $\psi_\kappa \circ \cos \in C^{2n}[-\pi, \pi]$. In this case Theorem 16 gives an approximation order of $h^{2n+\varepsilon-2t}$.

After this work was close to completion, we learned of the work by Levesley and Luo [15]. Using a different approach, their results show an approximation order of h^{n-t}, if $\psi_\kappa \circ \cos \in C^{2n}[-\pi, \pi]$. This is half the approximation order that we prove, although we believe that their result could be doubled by an analogous trick to the one that we employ. There are other notable differences between our results and those in [15]; we give our final error bounds for $\|u - s_u\|_t$ rather than $|L(u-s_u)(p)|$, $p \in S^{m-1}$, the dependence on f is explicit, we cover the more general case of non-invertible operators, and we do not require integer continuity of $\psi_\kappa \circ \cos$. On the other hand, the paper [15] does not require that the data points become dense on the entire sphere, just in a neighborhood of the evaluation point p.

When we restrict to the special case of Lagrange interpolation (for which L is the identity operator and $t = 0$), Theorem 16 gives an approximation order of $h^{2n+\varepsilon}$ for a shape function satisfying (30). This compares favorably with the result in [14] that gives an approximation order of $h^{n+\varepsilon-(m-1)/2}$, and also with the paper [12], which demonstrates an approximation order of h^n (since (30) implies, by a derivative formula for P_k in [21], $\psi_\kappa \in C^n[1-\varepsilon, 1]$, for some $0 < \varepsilon < 1$, which is the required assumption in [12]).

Assuming decay conditions on the $b_k(\psi_\kappa)$, rather than an integer continuity condition on the shape function ψ_κ, is of additional benefit. For example, consider Wendland's compactly supported C^{2n} kernels restricted to S^2 (as described in [6]), for which $\psi_\kappa \circ \cos \in C^{2n}[-\pi, \pi]$. Assuming that the order found in [15] can be doubled, one obtains an approximation order of h^{2n}. However, it seems that the $b_k(\psi_\kappa)$ have decay $O((1+k)^{-2n-2})$, and hence Theorem 16 gives an approximation order of h^{2n+1}. By using the decay of the coefficients $b_k(\psi_\kappa)$, rather than the continuity of ψ_κ, we seem to have squeezed out an extra power of h.

If the $b_k(\psi_\kappa)$ decay exponentially, one can show that $\psi_\kappa \in C^\infty[-1, 1]$. A consequence of the fast decay is, by part (b) of Theorem 16, that the convergence of the collocation method is spectral, which means that the error of approximation tends to zero faster than any power of h.

We conclude the paper with a collection of remarks.

Remark 17. Let us make a few comments about solving the system of Eqs. (28). Since $s_{\bar{u}, \Lambda, \kappa}$ is written in the form

$$s_{\bar{u}, \Lambda, \kappa} = \sum_{q \in Q} c_q L^q \kappa(\cdot, q).$$

Eqs. (28) lead to a linear system with matrix

$$A = (\kappa_L(p, q))_{p, q \in Q}.$$

To ensure that the coefficients c_q are uniquely determined, we need that A is invertible (note that although $s_{\bar{u}, \Lambda, \kappa}$ is unique, the c_q may not be). This is typically achieved by requiring that A is positive definite, which is determined solely by the positivity properties of the Legendre coefficients $b_k(\psi_{\kappa_L}) = L^\wedge(k)^2\, b_k(\psi_\kappa)$. The search for necessary and sufficient conditions on $b_k(\psi_{\kappa_L})$ that guarantee this is still ongoing (see [6] for a discussion). A simple *sufficient* condition for A to be positive definite is that only finitely many of the coefficients $b_k(\psi_{\kappa_L})$ are zero [24]. This condition is satisfied if we start with a kernel κ, for which $b_k(\psi_\kappa) > 0$, $k \geqslant 0$, and restrict ourselves to pseudodifferential operators for which $K_0(L)$ is finite. Unfortunately, the assumption $b_k(\psi_\kappa) > 0$ does not allow the use of initial kernels κ that are *conditionally* strictly positive definite, for example, multiquadrics [6]. In principle we could permit finitely many of the $b_k(\psi_\kappa)$ to be zero and still guarantee that A is positive definite, but then our assumption $f \in \mathscr{H}_{\kappa_L * \kappa_L}$ would be too restrictive. The condition $f \in \mathscr{H}_{\kappa_L * \kappa_L}$ presumes that f is missing the harmonics of degree k for which $b_k(\psi_\kappa) = 0$ and $L^\wedge(k) \neq 0$ (as well as the harmonics for which $L^\wedge(k) = 0$). Typically, this would be an unreasonable presumption. It is a topic of future research to see if we can reasonably employ kernels that do not satisfy $b_k(\psi_\kappa) > 0$, $k \geqslant 0$.

Remark 18. In applications we will usually be given the pseudodifferential operator L of some order t and a driving function f with certain smoothness, say $f \in H^r$. An important question is how to choose the kernel κ. There are two competing requirements. The kernel κ must be smooth enough so that (5) and (15) hold, but not so smooth that we no longer have $f \in \mathscr{H}_{\kappa_L * \kappa_L}$. On closer inspection, we can see that these two requirements are satisfied if there exist positive constants B_1 and B_2, and $\alpha_2 > \max\{0, 2t\}$ such that

$$B_1 (1+k)^{-1-\alpha_1} \leqslant b_k(\psi_\kappa) \leqslant B_2 (1+k)^{-1-\alpha_2},$$

for all k such that $b_k(\psi_\kappa) \neq 0$, where $\alpha_1 = r + 2t + 1 - m$. Moreover, when $b_k(\psi_\kappa)$ has the above decay, there exist positive constants B_1' and B_2' such that

$$B_1' \, \|f\|_{r+\alpha_2-\alpha_1} \leqslant \|f\|_{\kappa_L * \kappa_L} \leqslant B_2' \, \|f\|_r.$$

Ideally we should choose κ as smooth as possible, whilst still satisfying the requirements of the theorem. Note that in general, the smoother the driving function f, the smoother we can choose κ to be, and the better the predicted convergence. On the other hand, if $\alpha_1 < \alpha_2$, then the function f is not smooth enough in which case it may not be possible to choose an appropriate κ that would give rise to the approximation order stated in Theorem 16. However, in that case we may still get half of the approximation order if $f \in \mathscr{H}_\kappa$, which would occur whenever $r > (m-1)/2 - \min\{0, 2t\}$.

Remark 19. If n is a nonnegative integer such that $t > n + (m-1)/2$, then using the Sobolev Imbedding Theorem we can rewrite Theorem 16 with the error measured in the norm of $C^n(S^{m-1})$ rather than in the Sobolev norm $\|\cdot\|_t$. In particular, if $n = 0$, then we can bound the pointwise error.

Remark 20. With small adjustments, the results of this paper can be shown to hold for operators of a more general form than (13). Suppose that each frequency $k \geqslant 0$ is associated with an orthogonal matrix $O^k := (o^k_{lj})_{1 \leqslant l, j \leqslant N(k)}$. For any sequence $(L\hat{\ }(k))_{k \geqslant 0}$ and any $u \in H^s$, we can define an operator L by

$$Lu = \sum_{k=0}^{\infty} \sum_{l=1}^{N(k)} L\hat{\ }(k)\, \hat{u}_{kl} \sum_{j=1}^{N(k)} o^k_{lj} Y_{kj}. \tag{31}$$

Clearly, if all O^k are the identity matrices, then (31) reduces to (13). An example of the operator of the form (31) is

$$L = \frac{\partial^{2n+1}}{\partial \theta^{2n+1}}, \qquad n \geqslant 0,$$

defined on the circle S^1, where θ is the polar angle. In this case we have $Y_{k1} = \cos(k\theta)$, $Y_{k2} = \sin(k\theta)$, $L\hat{\ }(k) = k^{2n+1}$, and

$$O^k = \begin{pmatrix} 0 & (-1)^{n+1} \\ (-1)^n & 0 \end{pmatrix}, \qquad k \geqslant 0.$$

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Note added in proof. Since the submission of this paper, one of us has improved the error bounds of Theorem 16 [T. M. Morton, Improved error bounds for solving pseudodifferential equations on spheres by collocation with zonal kernels, *in* "Trends in Approximation Theory" (K. Kopotun, T Lyche, and M. Neamtu, Eds.), pp. 317–326, Vanderbilt University Press, Nashville, TN, 2001].

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